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Further Studies in Aesthetic Field Theory V; Higher Dimensions

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Abstract

We consider higher dimensional universes in which the universe is constructed from four-dimensional subuniverses. Independent parameters are introduced for the subuniverses. We then consider a small coupling between the subuniverses. Emphasis is placed on the eight-dimensional case. We find that the trends from the computer are in line with the natural boundary conditions $\Gamma^i_{jk} \to 0$ being satisfied. We do not find any significant improvement resulting as a consequence of the higher dimensions over and beyond the four-dimensional work we have done in previous papers.

1. *Introduction*

There has never been any satisfactory explanation of why the universe can be described within the framework of four dimensions. This has led many authors to investigate the consequences of higher dimensional theories beginning with Kaluza (1921). The emphasis has been to a large degree on five-dimensional theories as this is the simplest extension beyond four dimensions. A problem in any higher dimensional theory is to build in the apparent four-dimensional character of the universe, as appears empirically to be the case.

If one has a four-dimensional universe, one might wonder why there are not more such universes around. This could be accommodated in a higher dimensional theory of the type $4 \oplus 4 \oplus 4 \oplus 4 \oplus \ldots$. That is, one could have many independent four-dimensional universes. If this were the case, then it would appear to any observer that the universe is four-dimensional. Now, suppose instead of the 'side by side'† subuniverses being independent, there is a small coupling between them. Even though we would expect the essential four-

† A problem with five dimensions is that even though we would have four-dimensional subuniverses stacked next to each other along the fifth direction, the four-dimensional subuniverse would differ from the neighboring subuniverses by an infinitesimal amount. Thus, the different four-dimensional subuniverses would not be basically independent as they are in the approach we have taken.

dimensional character to be preserved, some effects due to higher dimension would be introduced into the four-dimensional subuniverses.

What might these effects be? There are some features that come to mind that could result from this coupling to higher dimensions. For example, it is possible that there be a creation of matter in our four-dimensional subuniverse arising from the higher dimensional 'source'. Such continuous creation has been postulated in cosmological theories. Another possible effect from the interrelation with the higher dimensions could be the presence of what appears to be a stochastic force affecting the motion of a particle. Such forces have been suggested as giving rise to quantum phenomena (Nelson, 1966; de la Pena-Auerbach, 1967).

Thus, it may be interesting to explore a universe with dimension $4n$, with n an integer. It seems unreasonable that n should be infinite. At the moment we have no *a priori* argument to decide what n should be. We shall consider in this paper the case of $n = 2$. This would be the simplest extension beyond the conventional four-dimensional world. It is not so unreasonable that a doubling of dimensions may yet turn out to be relevant. For example, it has never been understood why there appears to be so much more matter than antimatter in our own four-dimensional universe. Perhaps, it may be that what we need is a universe and an antiuniverse which have some coupling between them.

In this paper, we shall discuss an eight-dimensional universe consisting of two four-dimensional subuniverses which are coupled together. So far as we know such a structure has not been studied previously. It is of interest to see what definitive changes occur as a consequence of the additional four dimensions. Comparision will be made with computer studies when the additional four dimensions are not present.

2. *Aesthetic Field Theory*

In a series of papers (Muraskin, 1973a, 1973b; Muraskin & Ring, 1973)† we have been considering a field theory based on aesthetic mathematical ideas. We have demonstrated the existence of a bounded particle. There is at the same time, no sign of singularities appearing anywhere. The results are also consistent with a natural set of boundary conditions at infinity.

In all instances but one, the way we obtained a bounded particle was to assume that the underlying data ($\Gamma^{\alpha}_{\beta\gamma}, g_{\alpha\beta}$) is invariant under three-dimensional rotations at some origin point. The one exception to this was data 4 of Muraskin (1974). However, even in this case, it has not been proved that this data could not be obtained from a coordinate transformation on some other data which has the invariance property.

The problem with our invariant type data was that after a long enough computer run all the field components appear monotonically to approach zero. If this situation were to continue, we would not have a universe with

† Further references are found there.

many particles in it. Now, it is possible that longer computer runs would eventually show up additional structure, but it is also possible that this feature corresponds to a weakness of the theory as it is presently constructed.

In view of the difficulty in obtaining anything but a 'vacuum' far away from the particle, we may ask where is the notion of three-dimensional invariance of $\Gamma^{\alpha}_{\beta\gamma}, g_{\alpha\beta}$ leading us.

We may add that we have been unable to find a more general set of $\Gamma^{\alpha}_{\beta\gamma}$, $g_{\alpha\beta}$ satisfying the invariance requirement, beyond that suggested in Muraskin (1973b). Thus, it would appear that the basic alternatives would involve the continued running of the computer using the data in Muraskin (1973b), looking for more structure, or else we may have to give up the idea of such an invariance principle within aesthetic field theory.

However, there is still another alternative, We can go to higher dimensions. We know that the sixty-fourth Γ^{i}_{jk} associated with i, j, k running from one to four approach zero far down, say, the x^1-axis in Muraskin (1973b). But it is not clear that components like Γ^{1}_{25} which couple the two subuniverses should also have this property. That is, even though such components are small at the origin, their change from point to point is small, and so when we go far enough from the origin these coupling components may even become larger than the components associated with the four-dimensional subuniverses (having i, j, k going from one to four). Thus, their contribution to the four-dimensional subuniverses may become the dominant contribution and thus these coupling components may then actually act like a 'source' function. This could then lead to a reversal of monotonic behavior outside the particle.

We note that the effect of the coupling from higher dimensions on the four-dimensional subuniverses can be compared directly with the case of no coupling, using the computer, to see if the coupling is capable of leading to this desired effect.

3. 0(3) *Data*

We need to specify $g_{\alpha\beta}$, $\Gamma^{\alpha}_{\beta\gamma}$ and $e^{\alpha}{}_{i}$ for the eight-dimensional case. We chose $\Gamma^{\alpha}_{\beta\gamma}$ with α, β, γ running from one to four to be identical with the Γ^{i}_{jk} appearing in Muraskin (1973b) having $R^{i}{}_{jkl} \neq 0$. This data led to a maximum in $g_{44} \equiv g_{00}$ at the origin and satisfied the integrability equations. The underlying data (see equations (10), (11), (12), (13) of Muraskin (1973b)) is invariant under 0(3). The data for $\Gamma^{\alpha}_{\beta\gamma}$ with, α, β, γ running from five to eight was chosen to be identical with data 3 of Muraskin (1974). This also has an underlying 0(3) invariant structure and satisfied integrability. This data did not lead to a maximum or minimum in g_{44} at the origin. All other $\Gamma^{\alpha}_{\beta\gamma}$ (such as Γ^{1}_{25}, etc.) were chosen to be zero.

The data for $g_{\alpha\beta}$ with α, β running from one to four was taken to be the g_{ij} data of Muraskin (1973b) used in conjunction with the Γ^{i}_{jk} data there. For α, β running from five to eight we took the g_{ij} data number 3 from Muraskin (1974). The remaining $g_{\alpha\beta}$ were chosen to be zero.

The eight-dimensional $e^{\alpha}{}_{i}$ were chosen to be

$e^1{}_1 = 1$	$e^1{}_2 = 0$	$e^1{}_3 = 0$	$e^1{}_4 = 0$
$e^2{}_1 = 0$	$e^2{}_2 = 1$	$e^2{}_3 = 0$	$e^2{}_4 = 0$
$e^3{}_1 = 0$	$e^3{}_2 = 0$	$e^3{}_3 = 1$	$e^3{}_4 = 0$
$e^4{}_1 = 0$	$e^4{}_2 = 0$	$e^4{}_3 = 0$	$e^4{}_4 = 1$
$e^5{}_1 = -{\cdot}64 \times 10^{-3}$	$e^5{}_2 = -{\cdot}74 \times 10^{-3}$	$e^5{}_3 = -{\cdot}43 \times 10^{-3}$	$e^5{}_4 = {\cdot}32 \times 10^{-3}$
$e^6{}_1 = {\cdot}57 \times 10^{-3}$	$e^6{}_2 = {\cdot}82 \times 10^{-3}$	$e^6{}_3 = {\cdot}92 \times 10^{-3}$	$e^6{}_4 = {\cdot}81 \times 10^{-3}$
$e^7{}_1 = -{\cdot}3 \times 10^{-3}$	$e^7{}_2 = -{\cdot}762 \times 10^{-3}$	$e^7{}_3 = -{\cdot}99 \times 10^{-3}$	$e^7{}_4 = {\cdot}76 \times 10^{-3}$
$e^8{}_1 = {\cdot}54 \times 10^{-3}$	$e^8{}_2 = {\cdot}67 \times 10^{-3}$	$e^8{}_3 = -{\cdot}86 \times 10^{-3}$	$e^8{}_4 = {\cdot}9 \times 10^{-3}$
$e^1{}_5 = {\cdot}9 \times 10^{-3}$	$e^1{}_6 = {\cdot}7 \times 10^{-3}$	$e^1{}_7 = {\cdot}8 \times 10^{-3}$	$e^1{}_8 = -{\cdot}7 \times 10^{-3}$
$e^2{}_5 = {\cdot}85 \times 10^{-3}$	$e^2{}_6 = -{\cdot}47 \times 10^{-3}$	$e^2{}_7 = -{\cdot}7 \times 10^{-3}$	$e^2{}_8 = -{\cdot}95 \times 10^{-3}$
$e^3{}_5 = -{\cdot}75 \times 10^{-3}$	$e^3{}_6 = {\cdot}8 \times 10^{-3}$	$e^3{}_7 = {\cdot}9 \times 10^{-3}$	$e^3{}_8 = -{\cdot}856 \times 10^{-3}$
$e^4{}_5 = {\cdot}68 \times 10^{-3}$	$e^4{}_6 = {\cdot}85 \times 10^{-3}$	$e^4{}_7 = {\cdot}74 \times 10^{-3}$	$e^4{}_8 = {\cdot}954 \times 10^{-3}$
$e^5{}_5 = 1$	$e^5{}_6 = 0$	$e^5{}_7 = 0$	$e^5{}_8 = 0$
$e^6{}_5 = 0$	$e^6{}_6 = 1$	$e^6{}_7 = 0$	$e^6{}_8 = 0$
$e^7{}_5 = 0$	$e^7{}_6 = 0$	$e^7{}_7 = 1$	$e^7{}_8 = 0$
$e^8{}_5 = 0$	$e^8{}_6 = 0$	$e^8{}_7 = 0$	$e^8{}_8 = 1$

Thus, if there were no coupling between 1, 2, 3, 4 and 5, 6, 7, 8 in $e^{\alpha}{}_{i}$, we would have two independent four-dimensional universes, and for each we would get the same results as described in our previous papers. In the case of no coupling between the subuniverses, components such as Γ^{5}_{67}, however, would not go to zero at $x^1 \to \infty$ but would keep the same value they had at $x^1 = 0$. Thus, we may anticipate Γ^{i}_{jk} goes to a nonzero constant at infinity. Therefore, it is not clear at this point that we can construct an eight-dimensional theory satisfying the natural boundary conditions. This is a problem we should look into after we introduce coupling between the subuniverses.

We have set up the eight-dimensional run in the same manner as described in Muraskin & Ring (1972). We used, initially, an IBM 360/40 computer. Here we were getting a hundred points calculated every 75 minutes. Then we switched to an IBM 370/135 where we obtained 100 points every 23 minutes. This contrasted with 100 points every 7 minutes we were getting in our four-dimensional work with the IBM 360/40.

The problem we shall investigate is what effects are implied by the coupling between the two subuniverses as described by the above $e^{\alpha}{}_{i}$.

4. *Computer Results*

We ran the computer out to $x = 33{,}000$, making this, by far, the longest run to date. We chose a variable grid so that the value of the corrected Γ^{i}_{jk}, at the neighboring point, minus the corrected Γ^{i}_{jk} using half the grid, at this same point, was kept less than 10^{-10} throughout the run (Muraskin & Ring, 1972).

The Γ^{i}_{jk} with i, j, k running from one to four behaved as described in

Muraskin (1973b). By the time we were at $x = 20$, all sixty-four were growing smaller in magnitude. They continued growing smaller in a monotonic fashion, so far as we could tell from the grid size we used, for the rest of the run. Thus, no new turnabout points showed up for these components. This was also the case for the four-dimensional run which we ran concurrently for the sake of comparison.

Next, we studied components of the type Γ^1_{15} with one index running between five and eight and the other indices running from one to four. There was a small number of turnabout points for some of these components (this result involving turnabout points was true for the other types of components as well). Γ^1_{15} started out at $\cdot 106 \times 10^{-2}$. At $x = 33{,}000$ it was $-\cdot 95 \times 10^{-9}$. By the time we reached the end of the run just about all these components were decreasing in magnitude. Even though a few of these components were increasing their magnitude at the end of the run, we found, nevertheless, that all components of this type were many orders of magnitudes smaller than their starting values.

Components of the type Γ^1_{55} (with two indices running from five to eight remained essentially constant as expected from examination of the field equations, during the early part of the run. For example, Γ^1_{55} was $-\cdot 61 \times 10^{-2}$ at the origin. At $x = 169\cdot 2$ it was $-\cdot 60 \times 10^{-2}$. However, at $x = 33{,}000$ it had fallen off to $\cdot 93 \times 10^{-5}$. At the end of the run the large majority of components of the type Γ^1_{55} were decreasing in magnitude.

Components of the type Γ^8_{88}, for which all indices were in the range five to eight, tended to remain constant in the first part of the run as expected. Γ^8_{88} started out at $-1\cdot 07$. At $x = 19\cdot 19$ it was $-1\cdot 06$. At $x = 2020$ we were in the vicinity of a turnabout point with a value of $\cdot 33$. Its final value was $+\cdot 046$ at $x = 33{,}000$. Again most of the components of this type were decreasing in magnitude at the end of the run.

In Table 1 we gave the results for Γ^1_{11}, Γ^1_{21}, Γ^1_{15}, Γ^1_{55}, Γ^8_{88} as a function of x. Our final value of x was 33,000. Note we could not run the computer with much reliability much farther than this since, at $x = 33{,}000$ we started picking up components with magnitudes close to 10^{-10}. 10^{-10} is the maximum possible accuracy we can expect with the grid sizes we have been using.

In the beginning of the run, there were 162 components of Γ^i_{jk} that were increasing in magnitude. At the end of the run,† there were only 10.‡ The trend towards less components getting larger in magnitude is not monotonic as a result of the existence of turnabout points for the different components. We note that often in our previous work (Muraskin, 1971) we were accustomed to field components getting bigger in magnitude. Thus, our decrease in the number of increasing components from 162 to 10 is all the more striking. Note also that the magnitude of all but three components at $x = 33{,}000$ were smaller than the corresponding components at the origin.§

† These numbers were obtained by comparing the components at the beginning and end of a 100 point run at both the origin and at $x = 30{,}000$.

‡ We ran the computer to $x = 165{,}000$ even though some components were less than 10^{-10}. At this point none of the components were increasing in magnitude.

§ At $x = 54{,}000$ all components were smaller than their original value.

Even though it is not clear that the present trends need necessarily continue with still more accurate and longer runs, the impression that we get is that $\Gamma^i_{jk} \to 0$ for large x is not an unreasonable extrapolation from our present work.

If there were no coupling between the subuniverses, we would expect $\Gamma^i_{jk} \to A^i_{jk}$ at infinity with A^i_{jk} constant. However, even a small coupling, we see, suggests that A^i_{jk} may well be zero. This would constitute a set of natural boundary conditions.

Next, we compared our eight dimension results for Γ^i_{jk} with i, j, k running from one to four with the four-dimensional run. At $x = 0$ there was a slight difference between these two sets of values due to the effect of the small coupling coefficients $e^\alpha{}_i$. The difference at the origin between the two sets was in the range 10^{-4} to 10^{-6}. However, at the end of the run, the difference between the two sets was no larger than 10^{-9}. Thus, the four-dimensional and eight-dimensional values tended to get closer together although, in general, not in a monotonic fashion. The difference between the two sets as we approached the end of the run tended to grow even closer.

In summary, we have found that the four-dimensional and eight-dimensional values for the sixty-four Γ^i_{jk} differed at the end of the run by an extraordinarily small amount.

We have also repeated the calculation after increasing the coupling $e^\alpha{}_i$ by a factor of 100. Also, we tried a run employing data of the type used in Muraskin (1973a). In both cases we obtained results similar to that which we have described previously in this paper.

TABLE 1. Representative components as a function of x

x	Γ^1_{11}	Γ^1_{21}	Γ^1_{15}	Γ^1_{55}	Γ^8_{88}
0	1·06	2·06	$\cdot 11 \times 10^{-2}$	$-\cdot 61 \times 10^{-2}$	−1·07
·646	·25	·98	$\cdot 27 \times 10^{-3}$	$-\cdot 61 \times 10^{-2}$	−1·07
2·516	−·11	·22	$-\cdot 92 \times 10^{-4}$	$-\cdot 61 \times 10^{-2}$	−1·07
8·06	−·088	·03	$-\cdot 78 \times 10^{-4}$	$-\cdot 61 \times 10^{-2}$	−1·07
509·2	$-\cdot 19 \times 10^{-2}$	$\cdot 95 \times 10^{-5}$	$-\cdot 22 \times 10^{-5}$	$-\cdot 52 \times 10^{-2}$	−·44
1865	$-\cdot 5 \times 10^{-3}$	$\cdot 7 \times 10^{-6}$	$-\cdot 36 \times 10^{-6}$	$-\cdot 29 \times 10^{-2}$	·31
3600	$-\cdot 28 \times 10^{-3}$	$\cdot 19 \times 10^{-6}$	$-\cdot 97 \times 10^{-7}$	$-\cdot 46 \times 10^{-3}$	·30
11,400	$-\cdot 88 \times 10^{-4}$	$\cdot 19 \times 10^{-7}$	$-\cdot 85 \times 10^{-8}$	$-\cdot 15 \times 10^{-4}$	·13
33,000	$-\cdot 30 \times 10^{-4}$	$\cdot 22 \times 10^{-8}$	$-\cdot 95 \times 10^{-9}$	$\cdot 93 \times 10^{-5}$	·046

5. *Conclusions*

We have introduced higher dimensions in a way different from other authors. Each four-dimensional subuniverse is assigned parameters independently of the other four-dimensional subuniverses. We have emphasized the eight-dimensional case in this paper. Our computer runs were restricted to data for which the subuniverses had an underlying 0(3) invariant structure. The present work has not been successful in confirming our hopes that the higher dimensions would play a useful role in improving the results we had obtained in our four-dimensional studies. We foud no significant difference between the values of Γ^i_{jk} (i, j, k running from one to four) in the eight-dimensional run as compared with the four-dimensional run.

In a more positive vein, it appears that it may be possible to construct higher dimensional theories in which the subuniverses are coupled, such that the natural boundary conditions are satisfied.

In the next section, we will consider data not related to data invariant under O(3).

6. *Data in which all Invariants Vanish*

In Muraskin (1972) we found a solution to the integrability equations having the structure

$$\Gamma^\alpha_{\beta\gamma} = \delta^\alpha_\beta \phi_\gamma \tag{6.1}$$
$$\phi_\alpha \phi^\alpha = 0$$

With the use of the field equations we found that (6.1) led to a singular structure.

We can enlarge the data (6.1) as follows ($e_{\lambda\rho\beta\gamma}$ is the antisymmetric symbol which takes on the values $-1, 0, 1$)

$$\Gamma^\alpha_{\beta\gamma} = \delta^\alpha_\beta \phi_\gamma + g_{\beta\gamma}\psi^\alpha + \delta^\alpha_\gamma \theta_\beta + \sqrt{(-\det g_{\alpha\beta})} g^{\alpha\rho} B^\lambda \, e_{\lambda\rho\beta\gamma} \tag{6.2}$$

We find that when

$$g_{\alpha\beta} = \mathrm{diag}(-1, -1, -1, +1) \tag{6.3}$$

and

$$\begin{aligned} \psi_\alpha &= \phi_\alpha \\ \theta_\alpha &= A\phi_\alpha \\ B_\alpha &= B\phi_\alpha \end{aligned} \tag{6.4}$$

that the $R^i_{jkl} \neq 0$ integrability equations are satisfied provided

$$\phi_\alpha \phi^\alpha = 0 \tag{6.5}$$

Since $\Gamma^\alpha_{\beta\gamma}$ is constructed from a single independent null vector and from $g_{\alpha\beta}$ in (6.3), it follows that all invariants constructed from $\Gamma^\alpha_{\beta\gamma}$ and $g_{\alpha\beta}$ are zero. This is a necessary condition in order that $\Gamma^i_{jk} \to 0$ at infinity in a system where $g \equiv \det g_{ij} \neq 0$ at all points.

Equation (6.4) can be written as

$$\begin{aligned}
\theta_1 &= \phi_1 \frac{\theta_0}{\phi_0} & B_1 &= \phi_1 \frac{B_0}{\phi_0} \\
\theta_2 &= \phi_2 \frac{\theta_0}{\phi_0} & B_2 &= \phi_2 \frac{B_0}{\phi_0} \\
\theta_3 &= \phi_3 \frac{\theta_0}{\phi_0} & B_3 &= \phi_3 \frac{B_0}{\phi_0}
\end{aligned} \tag{6.6}$$

There are four vectors in the decomposition (6.2), but in order to satisfy integrability they must all be parallel at least at the origin. From (6.2) we can solve for $\theta_\alpha, \phi_\alpha, B_\alpha$. We get

$$\begin{aligned}
\theta_\alpha &= \frac{5\Gamma^\lambda_{\alpha\lambda} - 2\Gamma^\lambda_{\lambda\alpha}}{18} \\
\phi_\alpha &= \frac{4\Gamma^\lambda_{\lambda\alpha} - \Gamma^\lambda_{\alpha\lambda}}{18} \\
B^\alpha &= \tfrac{1}{6} g_{\beta\gamma} \Gamma^\gamma_{\lambda\chi} \frac{e^{\alpha\beta\lambda\chi}}{\sqrt{(-\det g_{\alpha\beta})}}
\end{aligned} \tag{6.7}$$

An $e^\alpha{}_i$ transformation will preserve the structure (6.2) and the resulting data will still obey the integrability equations. The $e^\alpha{}_i$ transformation leads to

$$\begin{aligned}
\Gamma^i_{jk} &= e_\alpha{}^i e^\beta{}_j, \ e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \\
g_{ij} &= e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} \\
\phi_i &= e^\alpha{}_i \phi_\alpha \\
\theta_i &= e^\alpha{}_i \phi_\alpha \\
B_i &= e^\alpha{}_i B_\alpha
\end{aligned} \tag{6.8}$$

$\sqrt{(-\det g_{\alpha\beta})}\, e_{\alpha\beta\gamma\delta}$ acts like a fourth-tank tensor under the $e^\alpha{}_i$ transformation. Thus, after an $e^\alpha{}_i$ transformation we have

$$\Gamma^i_{jk} = \delta^i_j \phi_k + g_{jk} \psi^i + \delta^i_k \theta_j + \sqrt{(-g)} g^{im} B^t e_{tmjk} \tag{6.9}$$

From (6.7) and $\Gamma^i_{jk;l} = 0, g_{ij;k} = 0$ we get for the change of θ_i, B_i, ϕ_i

$$\begin{aligned}
d\phi_i &= \Gamma^j_{ik} \phi_j \, dx^k \\
d\theta_i &= \Gamma^j_{ik} \theta_j \, dx^k \\
dB_i &= \Gamma^j_{ik} B_j \, dx^k
\end{aligned} \tag{6.10}$$

We used $\partial\sqrt{(-g)}/\partial x^l = \sqrt{(-g)}\Gamma^t_{tl}$ which follows from $g_{ij;k} = 0$. In $\Gamma^i_{jk;l} = 0$, $g_{ij;k} = 0$ theory, all vector functions of $\Gamma^i_{jk}, g_{ij}, \partial$, change according to

$$dA_i = \Gamma^j_{ik} A_j \, dx^k \tag{6.11}$$

We see that ϕ_i, θ_i, B_i ascribe to this law as expected. We next find that Γ^i_{jk} has the structure (6.9) at all points if it has it at one point. To prove this we note

$$\{\Gamma^i_{jk} - (\delta^i_j\phi_k + g_{jk}\psi^i + \delta^i_k\theta_j + \sqrt{(-g)}g^{im}B^t \, e_{tmjk})\}_{;l} = 0 \tag{6.12}$$

This follows since the individual terms obey an equation of the type

$$T^{ij\ldots}_{mn\ldots;l} = 0 \tag{6.13}$$

Expanding the semi-colon derivative and using the fact that (6.9) holds at the origin we get

$$\frac{\partial}{\partial x^l}\{\Gamma^i_{jk} - (\delta^i_j\phi_k + g_{jk}\psi^i + \delta^i_k\theta_j + \sqrt{(-g)}g^{im}B^t \, e_{tmjk})\} = 0 \tag{6.14}$$

In a similar fashion we see that all derivatives of the curly brackets vanish and thus the curly bracket is constant. Since it is zero, at the origin, it is then zero everywhere. Thus, Γ^i_{jk} has the structure (6.9) at all points.

We can also show that A and B in (6.4) are constants. From (6.8) we have $\theta_i = A\phi_i$ etc. Thus, $d\theta_i = A\, d\phi_i + dA\phi_i$. But, from (6.10) and (6.14) we see $dA = 0$.

From (6.7) we get

$$\frac{\theta_0}{\phi_0} = \frac{5\Gamma^t_{0t} - 2\Gamma^t_{t0}}{4\Gamma^t_{t0} - \Gamma^t_{0t}} \tag{6.15}$$

From (6.6) and (6.4) we have $A = \theta_0/\phi_0$. The quantity on the right side was calculated at various points down the x-axis by the computer. We found A to be constant to computer accuracy as expected from above.

We can now prove that the data (6.2) leads to singular structure. From (6.10) together with (6.9) we get, on using $\phi_i\phi^i = 0$

$$\frac{\partial\phi_i}{\partial x^k} = (1 + A)\phi_i\phi_k \tag{6.16}$$

This differential equation is not too different from the equation solved in Muraskin (1972) which had a singular structure. We can easily show that a singularity must develop from (6.16). Let us take $i = 1, k = 1$. Then we get

$$d\phi_1 = (1 + A)\phi_1^2 \, dx^1 \tag{6.17}$$

If $1 + A$ is positive when we proceed down the x-axis, we will be continually adding positive contributions to the function at the origin. Thus a singularity must eventually develop. If $1 + A$ is negative, the same situation will occur along the $-x$-axis.

Thus, in going from the data (6.1) to the more involved data (6.2), we still have not got around the problem of singularities.

On the other hand, in our previous work we had a four vector decomposition for which there was no sign of singularities developing anywhere in our computer work (Muraskin & Ring, 1972). On scrutiny, however, there are differences between this vector decomposition and (6.9). The vector in our previous work was not null. Also, there are minus sign differences. In Muraskin (1972) we pointed out that if we go to eight dimensions it is no longer a foregone conclusion that singularities will still develop. $\Sigma_{i=1}^{4} \phi_i \phi^i$ will no longer be zero if there is a coupling between the subuniverses. In eight dimensions $\Gamma^{\alpha}_{\beta\gamma}$ does not have the structure (6.2) since, for example, if $\phi_1 \neq 0$ then Γ^5_{51} would be non-zero if we had a vector decomposition. But we now have $\Gamma^5_{51} = 0$. The question is whether the coupling between the four-dimensional subuniverses can lead to damping effect. In the next section we shall discuss our computer results for this problem.

7. *Computer Results*

We have chosen $\Gamma^{\alpha}_{\beta\gamma}$, $g_{\alpha\beta}$, $e^{\alpha}{}_i$ as follows.

$\Gamma^{\alpha}_{\beta\gamma}$: α, β, γ running from one to four:

$$\phi_1 = \cdot 2 \qquad \phi_2 = \cdot 3 \qquad \phi_3 = \cdot 6 \qquad \phi_4 \equiv \phi_0 = \cdot 7$$
$$A = \tfrac{10}{7} \qquad B = \tfrac{3}{4}A \tag{7.1}$$

$g_{\alpha\beta} = \text{diag.}\,(-1, -1, -1, +1)$.

$\Gamma^{\alpha}_{\beta\gamma}$: α, β, γ running from five to eight:

$$\phi_1 = \cdot 2 \qquad \phi_2 = \cdot 3 \qquad \phi_3 = \cdot 6 \qquad \phi_4 = \cdot 7$$
$$B = 1 \qquad A = 0 \tag{7.2}$$

$g_{\alpha\beta} = \text{diag.}\,(+1, +1, +1, -1)$

All other $\Gamma^{\alpha}_{\beta\gamma}$, $g_{\alpha\beta}$ were chosen to be zero. We then required that g_{00} be a maximum or minimum at the origin by calculating $e^0{}_1$, $e^0{}_2$, $e^0{}_3$ in the manner described in Muraskin (1971). We chose $e^{\alpha}{}_i$ to be

$$\begin{array}{llll} e^1{}_1 = \cdot 9 & e^1{}_2 = -\cdot 13 & e^1{}_3 = -\cdot 187 & e^1{}_0 = -\cdot 34 \\ e^2{}_1 = \cdot 21 & e^2{}_2 = \cdot 5 & e^2{}_3 = -\cdot 24 & e^2{}_0 = \cdot 082 \\ e^3{}_1 = -\cdot 17 & e^3{}_2 = -\cdot 26 & e^3{}_3 = \cdot 65 & e^3{}_0 = -\cdot 163 \\ & & & e^0{}_0 = -\cdot 71 \end{array} \tag{7.3}$$

The data above describe two independent four-dimensional subuniverses. We next introduced coupling between the two subuniverses by means of eight-dimensional $e^{\alpha}{}_i$.

$$\begin{array}{llll} e^1{}_1 = 1 & e^1{}_2 = 0 & e^1{}_3 = 0 & e^1{}_4 = 0 \\ e^2{}_1 = 0 & e^2{}_2 = 1 & e^2{}_3 = 0 & e^2{}_4 = 0 \\ e^3{}_1 = 0 & e^3{}_2 = 0 & e^3{}_3 = 1 & e^3{}_4 = 0 \\ e^4{}_1 = 0 & e^4{}_2 = 0 & e^4{}_3 = 0 & e^4{}_4 = 1 \end{array} \qquad (7.4)$$

$$\begin{array}{llll} e^5{}_5 = \cdot 99 & e^5{}_6 = \cdot 97 & e^5{}_7 = \cdot 25 & e^5{}_8 = \cdot 47 \\ e^6{}_5 = 1 \cdot 2 & e^6{}_6 = -\cdot 34 & e^6{}_7 = -\cdot 27 & e^6{}_8 = \cdot 98 \\ e^7{}_5 = \cdot 24 & e^7{}_6 = \cdot 34 & e^7{}_7 = \cdot 76 & e^7{}_8 = -\cdot 96 \\ e^8{}_5 = -1 \cdot 8 & e^8{}_5 = \cdot 16 & e^8{}_7 = -\cdot 43 & e^8{}_8 = \cdot 86 \end{array} \qquad (7.5)$$

$$\begin{array}{llll} e^1{}_5 = \cdot 9 \times 10^{-3} & e^1{}_6 = \cdot 7 \times 10^{-3} & e^1{}_7 = \cdot 8 \times 10^{-3} & e^1{}_8 = -\cdot 7 \times 10^{-3} \\ e^2{}_5 = \cdot 85 \times 10^{-3} & e^2{}_6 = -\cdot 47 \times 10^{-3} & e^2{}_7 = -\cdot 7 \times 10^{-3} & e^2{}_8 = -\cdot 95 \times 10^{-3} \\ e^3{}_5 = -\cdot 75 \times 10^{-3} & e^3{}_6 = \cdot 8 \times 10^{-3} & e^3{}_7 = \cdot 91 \times 10^{-3} & e^3{}_8 = -\cdot 856 \times 10^{-3} \\ e^4{}_5 = \cdot 68 \times 10^{-3} & e^4{}_6 = \cdot 85 \times 10^{-3} & e^4{}_7 = \cdot 74 \times 10^{-3} & e^4{}_8 = -\cdot 954 \times 10^{-3} \end{array} \qquad (7.6)$$

$$\begin{array}{llll} e^5{}_1 = -\cdot 64 \times 10^{-3} & e^5{}_2 = -\cdot 74 \times 10^{-3} & e^5{}_3 = -\cdot 43 \times 10^{-3} & e^5{}_4 = \cdot 32 \times 10^{-3} \\ e^6{}_1 = \cdot 57 \times 10^{-3} & e^6{}_2 = \cdot 82 \times 10^{-3} & e^6{}_3 = \cdot 92 \times 10^{-3} & e^6{}_4 = \cdot 81 \times 10^{-3} \\ e^7{}_1 = -\cdot 3 \times 10^{-3} & e^7{}_2 = -\cdot 762 \times 10^{-3} & e^7{}_3 = -\cdot 99 \times 10^{-3} & e^7{}_4 = \cdot 76 \times 10^{-3} \\ e^8{}_1 = \cdot 54 \times 10^{-3} & e^8{}_2 = \cdot 67 \times 10^{-3} & e^8{}_3 = -\cdot 86 \times 10^{-3} & e^8{}_4 = \cdot 9 \times 10^{-3} \end{array} \qquad (7.7)$$

We ran this alongside the four-dimensional data given by (7.1) and (7.3). In our previous discussion based on 0(3) invariant data we found essentially no difference between the four-dimensional and eight-dimensional run so far as Γ^i_{jk}, i, j, k running from one to four. However, this was not the case here. The difference between the results for the four- and eight-dimensional run showed a definite increase in magnitude as we went down the x-axis, and the percentage increase was comparable to the percentage increase of a representative component.

The problem we encounter at this point is that the four-dimensional run we know will lead to singularities. When we considered the four-dimensional case on the computer we found many components were monotonically increasing suggesting a singularity was developing. There were a few components that had a turnabout point close to the origin. However, for the components that we graphed, we did not find any component having more than one turnabout point by the time we reached $x = \cdot 567$. The eight-dimensional run, although slightly different from the four-dimensional run, was still only different from the four-dimensional case for Γ^i_{jk} $i, j, k = 1$–4 in the second or third decimal place at $x = \cdot 567$. Thus, it became clear to us that if any damping was to occur it would involve an unfeasibly long run before we might hope to see it. Hence we decided to increase the coupling in (7.6) and (7.7) to make the four-dimensional and eight-dimensional runs differ by a substantial amount at the origin.

We decided to make the coupling terms so large that the coupling Γ^i_{jk} became even larger in many instances than Γ^i_{jk} i, j, k running for one to four. We then investigated whether a bound appeared. We chose to replace (7.6) and (7.7) with

$$\begin{matrix} e^1{}_5 = \cdot 19 & e^1{}_6 = -\cdot 4 & e^1{}_7 = \cdot 216 & e^1{}_8 = -\cdot 315 \\ e^2{}_5 = \cdot 12 & e^2{}_6 = -\cdot 24 & e^2{}_7 = \cdot 13 & e^2{}_8 = \cdot 36 \\ e^3{}_5 = -\cdot 19 & e^3{}_6 = \cdot 28 & e^3{}_7 = \cdot 21 & e^3{}_8 = -\cdot 16 \\ e^4{}_5 = -\cdot 11 & e^4{}_6 = \cdot 32 & e^4{}_7 = \cdot 222 & e^4{}_8 = \cdot 123 \end{matrix} \qquad (7.8)$$

$$\begin{matrix} e^1{}_5 = \cdot 19 & e^1{}_6 = -\cdot 34 & e^1{}_7 = \cdot 08 & e^1{}_8 = \cdot 38 \\ e^2{}_5 = \cdot 321 & e^2{}_6 = \cdot 425 & e^2{}_7 = -\cdot 254 & e^2{}_8 = -\cdot 159 \\ e^3{}_5 = \cdot 14 & e^3{}_6 = \cdot 251 & e^3{}_7 = -\cdot 3 & e^3{}_8 = \cdot 2 \\ e^4{}_5 = \cdot 085 & e^4{}_6 = -\cdot 16 & e^4{}_7 = -\cdot 135 & e^4{}_8 = -\cdot 342 \end{matrix} \qquad (7.9)$$

Unfortunately we did not find significant differences, so far as we could tell, between this run and the previous run which employed (7.7) and (7.8). There were somewhat more turnabout points for i, j, k running from one to four. But again, there was one or no turnabout point per component (we did find a component that had two turnabout points). We ran to $x = 1{\cdot}077$ and observed a trend toward runaway components. For example, we had

	Γ^3_{01}	Γ^3_{00}
$x = 0$	$-\cdot 82$	$\cdot 66$
$x = \cdot 3$	$-2{\cdot}21$	$\cdot 031$
$x = \cdot 606$	$-6{\cdot}51$	$4{\cdot}48$
$x = \cdot 975$	$-44{\cdot}47$	$62{\cdot}97$
$x = 1{\cdot}077$	$-112{\cdot}12$	$180{\cdot}64$

Taking differences of the field Γ^i_{jk} between the origin and $x = \cdot 06$ we found 404 components increasing in magnitude. In the region about $x = 1{\cdot}077$, 465 out of a possible 512 were increasing in magnitude. Note, this is the opposite kind of behavior that we obtained in the first part of this paper. This suggests the possibility that if we continue running we may well end up with all 512 components increasing in magnitude. It is never clear whether trends of this sort will eventually reverse themselves. However, we have not seen, thus far, any indication that a damping mechanism is at work.

8. *Conclusion*

Our approach to higher dimensions has, so far as we know, not been investigated previously. But from a practical point of view we have found no improvement in the results of our previous papers.

However, it may still be the case that our approach could have some validity. We have seen that the kind of results we obtained in this paper has been critically dependent on the form of the initial data. Four our 0(3) invariant data we found no significant difference for the values of Γ^i_{jk} (i, j, k = 1–4) in the eight-dimensional and four-dimensional cases after long runs down an axis. On the contrary, our data in Section 7 did not approach the four-dimensional results. Thus, it may be that there exists a set of as yet unknown data for which the higher dimensions may lead to some of the desirable effects discussed in Section 1.

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